

THE PHASE SPACE OF THE REDUCED THREE-WAVE INTERACTION DYNAMICAL SYSTEM

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ABSTRACT. We study the phase space of the reduced three-wave interaction dynamical system. Thanks to its phase space, we present a five-parameter family of modified reduced three-wave interaction dynamical systems. We give its symmetry and holomorphy conditions.

1. INTRODUCTION

The reduced three-wave interaction dynamical system is explicitly given by

$$(1) \quad \begin{cases} \frac{dx}{dt} = -2y^2 + \gamma x + \delta y + z, \\ \frac{dy}{dt} = 2xy - \delta x + \gamma y, \\ \frac{dz}{dt} = -2xz - 2z. \end{cases}$$

Here x, y, z denote unknown complex variables and δ and γ are complex constant parameters. In this paper, we study the phase space of (1) from the viewpoint of its accessible singularities and local index.

We show that all accessible singular points can be resolved if and only if

$$(2) \quad \{(\delta, \gamma) = (0, -1), (\delta, 0)\}.$$

Thanks to its phase space, we present a 5-parameter family of modified reduced three-wave interaction dynamical systems explicitly given by

$$(3) \quad \begin{cases} \frac{dx}{dt} = -2y^2 - (\alpha_1 + \alpha_3 - 2\alpha_5)y + z + \frac{\alpha_2 + \alpha_4 + 2(\alpha_1 + \alpha_3)\alpha_5}{2}, \\ \frac{dy}{dt} = 2xy - 2\alpha_5x + \sqrt{-1}(\alpha_1 - \alpha_3)y - \sqrt{-1}\frac{\alpha_2 - \alpha_4 + 2(\alpha_1 - \alpha_3)\alpha_5}{2}, \\ \frac{dz}{dt} = -2xz - (\alpha_2 + \alpha_4)x + \sqrt{-1}(\alpha_2 - \alpha_4)y - \sqrt{-1}(\alpha_1 - \alpha_3)z \\ \quad + \sqrt{-1}(\alpha_2\alpha_3 - \alpha_1\alpha_4). \end{cases}$$

Here x, y, z denote unknown complex variables and α_i ($i = 1, 2, \dots, 5$) are complex parameters.

We also give its symmetry and holomorphy conditions.

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2. ACCESSIBLE SINGULARITY AND LOCAL INDEX

Let us review the notion of *accessible singularity*. Let B be a connected open domain in \mathbb{C} and $\pi : \mathcal{W} \longrightarrow B$ a smooth proper holomorphic map. We assume that $\mathcal{H} \subset \mathcal{W}$ is a normal crossing divisor which is flat over B . Let us consider a rational vector field \tilde{v} on \mathcal{W} satisfying the condition

$$\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$

Fixing $t_0 \in B$ and $P \in \mathcal{W}_{t_0}$, we can take a local coordinate system (x_1, \dots, x_n) of \mathcal{W}_{t_0} centered at P such that $\mathcal{H}_{\text{smooth}}$ can be defined by the local equation $x_1 = 0$. Since $\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H}))$, we can write down the vector field \tilde{v} near $P = (0, \dots, 0, t_0)$ as follows:

$$\tilde{v} = \frac{\partial}{\partial t} + g_1 \frac{\partial}{\partial x_1} + \frac{g_2}{x_1} \frac{\partial}{\partial x_2} + \dots + \frac{g_n}{x_1} \frac{\partial}{\partial x_n}.$$

This vector field defines the following system of differential equations

$$(4) \quad \frac{dx_1}{dt} = g_1(x_1, \dots, x_n, t), \quad \frac{dx_2}{dt} = \frac{g_2(x_1, \dots, x_n, t)}{x_1}, \dots, \quad \frac{dx_n}{dt} = \frac{g_n(x_1, \dots, x_n, t)}{x_1}.$$

Here $g_i(x_1, \dots, x_n, t)$, $i = 1, 2, \dots, n$, are holomorphic functions defined near $P = (0, \dots, 0, t_0)$.

DEFINITION 2.1. With the above notation, assume that the rational vector field \tilde{v} on \mathcal{W} satisfies the condition

$$(A) \quad \tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$

We say that \tilde{v} has an *accessible singularity* at $P = (0, \dots, 0, t_0)$ if

$$x_1 = 0 \text{ and } g_i(0, \dots, 0, t_0) = 0 \text{ for every } i, \ 2 \leq i \leq n.$$

If $P \in \mathcal{H}_{\text{smooth}}$ is not an accessible singularity, all solutions of the ordinary differential equation passing through P are vertical solutions, that is, the solutions are contained in the fiber \mathcal{W}_{t_0} over $t = t_0$. If $P \in \mathcal{H}_{\text{smooth}}$ is an accessible singularity, there may be a solution of (4) which passes through P and goes into the interior $\mathcal{W} - \mathcal{H}$ of \mathcal{W} .

Here we review the notion of *local index*. Let v be an algebraic vector field with an accessible singular point $\vec{p} = (0, \dots, 0)$ and (x_1, \dots, x_n) be a coordinate system in a neighborhood centered at \vec{p} . Assume that the system associated with v near \vec{p} can be written as

$$(5) \quad \frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \frac{1}{x_1} \left\{ \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ a_{(n-1)1} & a_{(n-1)2} & \dots & a_{(n-1)(n-1)} & 0 \\ a_{n1} & a_{n2} & \dots & a_{n(n-1)} & a_{nn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} x_1 h_1(x_1, \dots, x_n, t) \\ h_2(x_1, \dots, x_n, t) \\ \vdots \\ h_{n-1}(x_1, \dots, x_n, t) \\ h_n(x_1, \dots, x_n, t) \end{pmatrix} \right\},$$

($h_i \in \mathbb{C}(t)[x_1, \dots, x_n]$, $a_{ij} \in \mathbb{C}(t)$)

where h_1 is a polynomial which vanishes at \vec{p} and $h_i, i = 2, 3, \dots, n$ are polynomials of order at least 2 in x_1, x_2, \dots, x_n . We call ordered set of the eigenvalues $(a_{11}, a_{22}, \dots, a_{nn})$ *local index* at \vec{p} .

We are interested in the case with local index

$$(6) \quad (1, a_{22}/a_{11}, \dots, a_{nn}/a_{11}) \in \mathbb{Z}^n.$$

These properties suggest the possibilities that a_1 is the residue of the formal Laurent series:

$$(7) \quad y_1(t) = \frac{a_{11}}{(t - t_0)} + b_1 + b_2(t - t_0) + \dots + b_n(t - t_0)^{n-1} + \dots \quad (b_i \in \mathbb{C}),$$

and the ratio $(1, a_{22}/a_{11}, \dots, a_{nn}/a_{11})$ is resonance data of the formal Laurent series of each $y_i(t)$ ($i = 2, \dots, n$), where (y_1, \dots, y_n) is original coordinate system satisfying $(x_1, \dots, x_n) = (f_1(y_1, \dots, y_n), \dots, f_n(y_1, \dots, y_n))$, $f_i(y_1, \dots, y_n) \in \mathbb{C}(t)(y_1, \dots, y_n)$.

If each component of $(1, a_{22}/a_{11}, \dots, a_{nn}/a_{11})$ has the same sign, we may resolve the accessible singularity by blowing-up finitely many times. However, when different signs appear, we may need to both blow up and blow down.

The α -test,

$$(8) \quad t = t_0 + \alpha T, \quad x_i = \alpha X_i, \quad \alpha \rightarrow 0,$$

yields the following reduced system:

$$(9) \quad \frac{d}{dT} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix} = \frac{1}{X_1} \begin{bmatrix} a_{11}(t_0) & 0 & 0 & \dots & 0 \\ a_{21}(t_0) & a_{22}(t_0) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ a_{(n-1)1}(t_0) & a_{(n-1)2}(t_0) & \dots & a_{(n-1)(n-1)}(t_0) & 0 \\ a_{n1}(t_0) & a_{n2}(t_0) & \dots & a_{n(n-1)}(t_0) & a_{nn}(t_0) \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix},$$

where $a_{ij}(t_0) \in \mathbb{C}$. Fixing $t = t_0$, this system is the system of the first order ordinary differential equation with constant coefficient. Let us solve this system. At first, we solve the first equation:

$$(10) \quad X_1(T) = a_{11}(t_0)T + C_1 \quad (C_1 \in \mathbb{C}).$$

Substituting this into the second equation in (9), we can obtain the first order linear ordinary differential equation:

$$(11) \quad \frac{dX_2}{dT} = \frac{a_{22}(t_0)X_2}{a_{11}(t_0)T + C_1} + a_{21}(t_0).$$

By variation of constant, in the case of $a_{11}(t_0) \neq a_{22}(t_0)$ we can solve explicitly:

$$(12) \quad X_2(T) = C_2(a_{11}(t_0)T + C_1)^{\frac{a_{22}(t_0)}{a_{11}(t_0)}} + \frac{a_{21}(t_0)(a_{11}(t_0)T + C_1)}{a_{11}(t_0) - a_{22}(t_0)} \quad (C_2 \in \mathbb{C}).$$

This solution is a single-valued solution if and only if

$$\frac{a_{22}(t_0)}{a_{11}(t_0)} \in \mathbb{Z}.$$

In the case of $a_{11}(t_0) = a_{22}(t_0)$ we can solve explicitly:

$$(13) \quad X_2(T) = C_2(a_{11}(t_0)T + C_1) + \frac{a_{21}(t_0)(a_{11}(t_0)T + C_1)\text{Log}(a_{11}(t_0)T + C_1)}{a_{11}(t_0)} \quad (C_2 \in \mathbb{C}).$$

This solution is a single-valued solution if and only if

$$a_{21}(t_0) = 0.$$

Of course, $\frac{a_{22}(t_0)}{a_{11}(t_0)} = 1 \in \mathbb{Z}$. In the same way, we can obtain the solutions for each variables (X_3, \dots, X_n) . The conditions $\frac{a_{jj}(t)}{a_{11}(t)} \in \mathbb{Z}$, $(j = 2, 3, \dots, n)$ are necessary condition in order to have the Painlevé property.

3. CONSTRUCTION OF THE PHASE SPACE

In order to consider the phase space for the system (1), let us take the compactification $[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3$ of $(x, y, z) \in \mathbb{C}^3$ with the natural embedding

$$(x, y, z) = (z_1/z_0, z_2/z_0, z_3/z_0).$$

Moreover, we denote the boundary divisor in \mathbb{P}^3 by \mathcal{H} . Extend the regular vector field on \mathbb{C}^3 to a rational vector field \tilde{v} on \mathbb{P}^3 . It is easy to see that \mathbb{P}^3 is covered by four copies of \mathbb{C}^3 :

$$\begin{aligned} U_0 &= \mathbb{C}^3 \ni (x, y, z), \\ U_j &= \mathbb{C}^3 \ni (X_j, Y_j, Z_j) \quad (j = 1, 2, 3), \end{aligned}$$

via the following rational transformations

$$\begin{aligned} X_1 &= 1/x, & Y_1 &= y/x, & Z_1 &= z/x, \\ X_2 &= x/y, & Y_2 &= 1/y, & Z_2 &= z/y, \\ X_3 &= x/z, & Y_3 &= y/z, & Z_3 &= 1/z. \end{aligned}$$

The following Lemma shows that this rational vector field \tilde{v} has seven accessible singular points on the boundary divisor $\mathcal{H} \subset \mathbb{P}^3$.

LEMMA 3.1. *The rational vector field \tilde{v} has four accessible singular points:*

$$(14) \quad \begin{cases} P_1 = \{(X_1, Y_1, Z_1) | X_1 = Y_1 = Z_1 = 0\}, \\ P_2 = \{(X_1, Y_1, Z_1) | X_1 = Z_1 = 0, Y_1 = \sqrt{-1}\}, \\ P_3 = \{(X_1, Y_1, Z_1) | X_1 = Z_1 = 0, Y_1 = -\sqrt{-1}\}, \\ P_4 = \{(X_3, Y_3, Z_3) | X_3 = Y_3 = Z_3 = 0\}, \end{cases}$$

where the point P_4 has multiplicity of order 4.

This lemma can be proven by a direct calculation. \square

Next let us calculate its local index at each point.

Singular point	Type of local index
P_1	$(0, 2, -2)$
P_2	$(-2, -4, -4)$
P_3	$(-2, -4, -4)$

In order to do analysis for the accessible singular point P_4 , we need to replace a suitable coordinate system because this point has multiplicity of order 4.

At first, let us do the Painlevé test. To find the leading order behaviour of a singularity at $t = t_1$ one sets

$$\begin{cases} x \propto \frac{a}{(t - t_1)^m}, \\ y \propto \frac{b}{(t - t_1)^n}, \\ z \propto \frac{c}{(t - t_1)^p}, \end{cases}$$

from which it is easily deduced that

$$m = 1, \quad n = 0, \quad p = 2.$$

Each order of pole (m, n, p) suggests a suitable coordinate system to do analysis for the accessible singular point P_2 , which is explicitly given by

$$(X, Y, Z) = \left(\frac{1}{x}, y, \frac{z}{x^2} \right).$$

In this coordinate, the singular points are given as follows:

$$\begin{cases} P_4^{(1)} = \left\{ (X, Y, Z) = \left(0, \frac{\delta}{2}, 0 \right) \right\}, \\ P_4^{(2)} = \left\{ (X, Y, Z) = \left(0, \frac{\delta}{2}, -1 \right) \right\}. \end{cases}$$

Next let us calculate its local index at each point.

Singular point	Type of local index
$P_4^{(1)}$	$(0, 2, -2)$
$P_4^{(2)}$	$(1, 2, 2)$

Now, we try to resolve the accessible singular point $P_4^{(2)}$.

Step 0: We take the coordinate system centered at $P_4^{(2)}$:

$$p = X, \quad q = Y - \frac{\delta}{2}, \quad r = Z + 1.$$

In this coordinate, the system (1) is rewritten as follows:

$$\frac{d}{dt} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \frac{1}{p} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \dots \right\}.$$

By considering the ratio $(1, \frac{2}{1}, \frac{2}{1}) = (1, 2, 2)$, we obtain the resonances $(2, 2)$. This property suggests that we will blow up two times to each direction $q^{(1)}$ (resp. $r^{(1)}$).

Step 1: We blow up at the point $P_2^{(2)}$:

$$p^{(1)} = p, \quad q^{(1)} = \frac{q}{p}, \quad r^{(1)} = \frac{r}{p}.$$

Step 2: We blow up at the point $P_4 = \{(p^{(1)}, q^{(1)}, r^{(1)}) = (0, 0, 0)\}$:

$$u = p^{(1)}, \quad v = \frac{q^{(1)} + \frac{\delta\gamma}{2}}{p^{(1)}}, \quad w = \frac{r^{(1)} + 2(\gamma + 1)}{p^{(1)}}.$$

In this coordinate, the system (1) is rewritten as follows:

$$(15) \quad \begin{cases} \frac{du}{dt} = g_1(u, v, w), \\ \frac{dv}{dt} = \frac{\delta\gamma}{u} + g_2(u, v, w), \\ \frac{dw}{dt} = \frac{\gamma(\gamma + 1)}{u} + g_3(u, v, w), \end{cases}$$

where $g_i(u, v, w) \in \mathbb{C}[u, v, w]$ ($i = 1, 2, 3$).

Each right-hand side of the system (15) is a *polynomial* if and only if

$$(16) \quad \begin{cases} \delta\gamma = 0, \\ \gamma(\gamma + 1) = 0. \end{cases}$$

These equations can be solved as follows:

$$(17) \quad \{(\delta, \gamma) = (0, -1), (\delta, 0)\}.$$

THEOREM 3.2. *Under the assumption (17), the phase space \mathcal{X} for the system (1) is obtained by gluing four copies of \mathbb{C}^3 :*

$$U_j = \mathbb{C}^3 \ni \{(x_j, y_j, z_j)\}, \quad j = 0, 1, 2, 3$$

via the following birational transformations:

$$(18) \quad \begin{aligned} 0) & \quad x_0 = x, \quad y_0 = y, \quad z_0 = z, \\ 1) & \quad x_1 = \frac{1}{x}, \quad y_1 = -(y - \sqrt{-1}x)x, \quad z_1 = zx, \\ 2) & \quad x_2 = \frac{1}{x}, \quad y_2 = -(y + \sqrt{-1}x)x, \quad z_2 = zx, \\ 3) & \quad x_3 = \frac{1}{x}, \quad y_3 = -\left(\left(y - \frac{\delta}{2}\right)x + \frac{\delta\gamma}{2}\right)x, \quad z_3 = z + x^2 + 2(\gamma + 1)x. \end{aligned}$$

We remark that these transition functions satisfy the condition:

$$dx_i \wedge dy_i \wedge dz_i = dx \wedge dy \wedge dz \quad (i = 1, 2, 3).$$

4. MODIFIED REDUCED THREE-WAVE INTERACTION DYNAMICAL SYSTEM

In this section, we present a 5-parameter family of modified reduced three-wave interaction dynamical systems explicitly given by

$$(19) \quad \begin{cases} \frac{dx}{dt} = -2y^2 - (\alpha_1 + \alpha_3 - 2\alpha_5)y + z + \frac{\alpha_2 + \alpha_4 + 2(\alpha_1 + \alpha_3)\alpha_5}{2}, \\ \frac{dy}{dt} = 2xy - 2\alpha_5x + \sqrt{-1}(\alpha_1 - \alpha_3)y - \sqrt{-1}\frac{\alpha_2 - \alpha_4 + 2(\alpha_1 - \alpha_3)\alpha_5}{2}, \\ \frac{dz}{dt} = -2xz - (\alpha_2 + \alpha_4)x + \sqrt{-1}(\alpha_2 - \alpha_4)y - \sqrt{-1}(\alpha_1 - \alpha_3)z \\ \quad + \sqrt{-1}(\alpha_2\alpha_3 - \alpha_1\alpha_4). \end{cases}$$

Here x, y, z denote unknown complex variables and α_i ($i = 1, 2, \dots, 5$) are complex parameters.

PROPOSITION 4.1. *This system is invariant under the following transformations:*

$$(20) \quad \begin{aligned} s(x, y, z; \alpha_1, \alpha_2, \dots, \alpha_5) \rightarrow & \left(x - \frac{\sqrt{-1}(\alpha_2 - \alpha_4)}{2(y - \alpha_5)}, y, \frac{1}{4(y - \alpha_5)^2} \{ 4y^2z - 8\alpha_5yz \right. \\ & + 4\sqrt{-1}(\alpha_2 - \alpha_4)xy - 4\sqrt{-1}(\alpha_2 - \alpha_4)\alpha_5x \\ & - 2(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)y + 4\alpha_5^2z \\ & \left. + (\alpha_2 - \alpha_4)(\alpha_2 - \alpha_4 + 2(\alpha_1 - \alpha_3)\alpha_5) \right); \\ & \alpha_1, \alpha_4, \alpha_3, \alpha_2, \alpha_5), \\ \pi(x, y, z; \alpha_1, \alpha_2, \dots, \alpha_5) \rightarrow & (x, -y, z; -\alpha_3, \alpha_4, -\alpha_1, \alpha_2, -\alpha_5), \end{aligned}$$

where the transformations s and π satisfy the relations:

$$(21) \quad s^2 = \pi^2 = 1, \quad (s\pi)^2 = 1.$$

THEOREM 4.2. *The phase space \mathcal{X} for the system (19) is obtained by gluing four copies of \mathbb{C}^3 :*

$$U_j = \mathbb{C}^3 \ni \{(x_j, y_j, z_j)\}, \quad j = 0, 1, 2, 3$$

via the following birational transformations:

$$\begin{aligned}
 (22) \quad & 0) \ x_0 = x, \quad y_0 = y, \quad z_0 = z, \\
 & 1) \ x_1 = \frac{1}{x}, \quad y_1 = -(y - \sqrt{-1}x + \alpha_1)x, \quad z_1 = (z + \alpha_2)x, \\
 & 2) \ x_2 = \frac{1}{x}, \quad y_2 = -(y + \sqrt{-1}x + \alpha_3)x, \quad z_2 = (z + \alpha_4)x, \\
 & 3) \ x_3 = \frac{1}{x}, \quad y_3 = -\left((y - \alpha_5)x - \frac{\sqrt{-1}(\alpha_2 - \alpha_4)}{2}\right)x, \\
 & \quad z_3 = z + x^2 + \sqrt{-1}(\alpha_1 - \alpha_3)x.
 \end{aligned}$$

These transition functions satisfy the condition:

$$dx_i \wedge dy_i \wedge dz_i = dx \wedge dy \wedge dz \quad (i = 1, 2, 3).$$

THEOREM 4.3. *Let us consider a system of first order ordinary differential equations in the polynomial class:*

$$\frac{dx}{dt} = f_1(x, y, z), \quad \frac{dy}{dt} = f_2(x, y, z), \quad \frac{dz}{dt} = f_3(x, y, z) \quad (f_i(x, y, z) \in \mathbb{C}[x, y, z]).$$

We assume that

(A1) $\deg(f_i) = 2$ with respect to x, y, z .

(A2) The right-hand side of this system becomes again a polynomial in each coordinate system (x_i, y_i, z_i) ($i = 1, 2, 3$).

Then such a system coincides with the system (19).

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